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ORIGINAL PAPER

The variational spline method for solving Troesch's problem

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Abstract In this paper we present a variational approximation method for solving Troesch's problem. The existence and the uniqueness of this problem are shown. Moreover, we construct a sequence of approximate solutions of the problem from the number of knots of a partition of the domain. Such sequence converges to the exact solution of the problem. Finally, we analyze some graphical and numerical examples in order to show the efficiency of our method.

Keywords Troesch's problem · EDO · Variational method · Cubic splines

Mathematics Subject Classification 65D05 · 65D07 · 65D10 · 65D17

1 Introduction

The non-linear two-point boundary value problem, Bratu's equation and Troech's problems, occur engineering and science, and they may be used to model some chemical reaction–diffusion and heat transfer processes. In [1] the authors obtained an

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analytical expression pertaining to the concentration of substrate using Homotopy perturbation method for all values of the parameters. Mathematical modeling for some chemical reactions, that are usually accompanied with mass and energy transfer, is based on material and energy balance. One can generate a set of differential equations known as the reaction–diffusion problem. The reaction–diffusion equations are paid more attention in analyzing and designing chemical and catalytic reactors [7]. The same phenomena exists in electrochemical processes, with the added complexity of a varying potential field, and considerable research has been done for electrochemical reactions occurring in the porous electrode [8].

Different techniques for the construction of a curve have been developed in recent years, for example interpolation by spline functions, based on the minimization of a certain functional in an adequate Sobolev subspace (see [4–6]). Such a functional may represent minimal energy, or some physical considerations such as minimization of the air of a surface, or minimization of the curvature or variation of curvature (see [4]). These techniques have many applications in CAD, CAGD and Earth Sciences.

In [2], the authors considered the boundary-value problem, called Troesch's problem

$$u'' = \lambda \sinh(\lambda u),\ 0 \leq x \leq 1,$$

with boundary conditions

$$u(0) = 0,\ u(1) = 1.$$

They applied the reproducing kernel method for solving such problem. They used numerical examples to illustrate the accuracy and implementation of the method. The analytical result of the equation has been obtained in terms of a convergent series with easily computable components. This type of problems was also described and solved by Weibel in [11]. Later in 1976, Troesch in [10] found its numerical solution by the shooting method. Recently, this problem has been studied extensively. Troesch's equation appears in engineering and science, including the modeling of chemical reaction–diffusion and heat transfer processes.

A variational method is proposed in this work in order to solve Troesch's problem in a space of the B-spline functions. The solution is obtained by resolving a sequence of boundary-value problems in some spaces of B-splines functions. We study some characterizations of these functions, and we shall express them as some linear combinations of the B-spline basis functions. Under adequate hypotheses, we prove that such sequence converges to the exact solution of the problem. We present some graphical and numerical examples in order to show the efficiency of our method.

The remainder of this paper is organized as follows. In Sect. 2, after briefly recall on some preliminaries and notations we formulate the problem. Section 3 is devoted to study how to construct and to compute some sequence of solutions of the problem from the number of knots in the partition of the domain. In sect. 4 we prove that such sequence converges to the exact solution of the problem. Finally, in sect. 5 we present some numerical examples to illustrate the method.

2 Preliminaries and formulation of the problem

For any $n \in \mathbb{N}^*$ and given $a, b \in \mathbb{R}$ with $a < b$, we consider the interval $I = (a, b)$ and we denote by $\mathbb{P}_n(I)$ the linear space of the real polynomials with a degree less than or equal to n.

Now, let $H^1(I)$ be the usual Sobolev space of (classes of) functions u belonging to $L^2(I)$, together with the derivative $u^{(1)}$, in the distribution sense. This space is equipped with the inner semi-products

$$(u, v)_\ell = \int_I u^{(\ell)}(x)\, v^{(\ell)}(x) dx,\ 0 \le \ell \le 1,$$

being $u^{(0)} = u, u^{(1)} = u'$ and $u^{(\ell)}$ the derivative function of order ℓ, the corresponding semi-norms $|u|_\ell = ((u, u)_\ell)^{1/2}\ ,\ 0 \le \ell \le 1$ and the norm $\|u\| = \left(\sum_{\ell \le 1} |u|_\ell^2\right)^{1/2}$.

We denote by $\|u\|_0 = \left(\int_I u(x)^2 dx\right)^{1/2}$ the norm of $L^2(I)$.

Let $T_n = \{x_0, \ldots, x_n\}$ be a subset of distinct points of $[a, b]$, with $x_i = a + i\dfrac{b-a}{n}$, $i = 0, \ldots, n$. We denote by $S_3^2(T_n)$ the space of splines functions of degree 3 and class 2 constructed over the partition T_n, that is, $S_3^2(T_n)$ is given by

$$S_3^2(T_n) = \left\{s \in C^2[a, b] \mid s|_{([x_{i-1}, x_i])} \in \mathbb{P}_3([x_{i-1}, x_i]),\ i = 1, \ldots, n\right\}.$$

It is known that $\dim S_3^2(T_n) = n + 3$.

Now, let $\{B_i^n : i = 1, \ldots, n + 3\}$ be the basis functions of B-splines of $S_3^2(T_n)$ constructed from the knots

$$x_{-2} = x_{-1} = x_0 < x_1 < \ldots < x_n = x_{n+1} = x_{n+2}.$$

It is verified that $S_3^2(T_n)$ is a subset of $H^2(I)$.

Let $f \in L^2(I \times I)$ and F be the functional defined in $H^2(I)$ by

$$Fu(x) = f(x, u(x)),\ x \in I.$$

For each $u \in H^1(I)$, we have that Fu belongs to $L^2(I)$.

We consider the problem: Find $\sigma \in C^2(I)$ such that

$$\begin{cases} L\sigma = F\sigma \text{ in } I \\ \sigma(a) = \alpha,\ \sigma(b) = \beta \end{cases} \tag{1}$$

being $Lu = u''$ for all $u \in C^2(I)$ and $\alpha,\ \beta \in \mathbb{R}$.

3 Computing the approximating solution

We are going to show in this section how to construct the solution of our problem. To this end, let $\sigma_0 = 1$ and $\sigma_1 \in S_3^2(T_1)$ be the solution of the problem: Find $\sigma_1 \in S_3^2(T_1)$ and $(\lambda_1^1, \lambda_2^1) \in \mathbb{R}^2$ such that

$$\begin{cases} -\displaystyle\int_a^b \sigma_1'(x)\, u'(x)dx + \lambda_1^1 u(a) + \lambda_2^1 u(b) = \int_a^b f(x, \sigma_0(x))\, u(x)dx, \; \forall u \in S_3^2(T_1), \\ \sigma_1(a) = \alpha, \; \sigma_1(b) = \beta \end{cases}$$

Taking $\sigma_1(x) = \displaystyle\sum_{i=1}^{4} c_i B_i^1(x)$, by linearity we have that

$$-\sum_{i=1}^{4} \left(\int_a^b (B_i^1)'(x)\, (B_j^1)'(x)dx \right) c_i + B_j^1(a)\, \lambda_1^1 + B_j^1(b)\, \lambda_2^1 = \int_a^b f(x, 1)\, B_j^1(x)dx,$$

for $j = 1, \ldots, 4,$ and $\displaystyle\sum_{i=1}^{4} c_i B_i^1(a) = \alpha, \; \sum_{i=1}^{4} c_i B_i^1(b) = \beta.$

This is a linear system with unknowns $c_1, \ldots, c_4, \lambda_1^1, \lambda_2^1$ that, in matrix form, it can be expressed as

$$\begin{pmatrix} A_1 & D_1^t \\ D_1 & 0 \end{pmatrix} \begin{pmatrix} C_1^t \\ \lambda_1^t \end{pmatrix} = \begin{pmatrix} b_1^t \\ \mu^t \end{pmatrix}$$

being

$$\begin{aligned} A_1 &= \left(-\int_a^b (B_i^1)'(x)\, (B_j^1)'(x)dx \right)_{\substack{1 \le i \le 4 \\ 1 \le j \le 4}}; \\ D_1 &= \left(B_j^1(a_i) \right)_{\substack{i=1,2 \\ j=1,\ldots,4}} \text{ with } a_1 = a, \; a_2 = b; \\ C_1 &= (c_1, \ldots, c_4); \\ \lambda_1 &= \left(\lambda_1^1, \lambda_2^1 \right); \\ b_1 &= \left(\int_a^b f(x, 1)\, B_j^1(x)dx \right)_{j=1,\ldots,4}; \\ \mu &= (\alpha, \beta) \end{aligned}$$

Reasoning by induction, we assume that the approximation $\sigma_{n-1} \in S_3^2(T_{n-1})$, $\sigma_{n-1}(a) = \alpha$ and $\sigma_{n-1}(b) = \beta$, is constructed and we look for $\sigma_n \in S_3^2(T_n)$ and $(\lambda_1^n, \lambda_2^n) \in \mathbb{R}^2$ such that

$$\begin{cases} -\displaystyle\int_a^b \sigma_n'(x)\, u'(x)dx + \lambda_1^n u(a) + \lambda_2^n u(b) = \int_a^b f(x, \sigma_{n-1}(x))\, u(x)dx, \; \forall u \in S_3^2(T_n) \\ \sigma_n(a) = \alpha, \; \sigma_n(b) = \beta. \end{cases} \tag{2}$$

Hence, $\sigma_n(x) = \sum_{i=1}^{n+3} c_i B_i^n(x)$ and by reasoning as to compute σ_1, we obtain a linear system with unknowns $c_1, \ldots, c_{n+3}, \lambda_1^n, \lambda_2^n$ that, in matrix form, it can be expressed by

$$\begin{pmatrix} A_n & D_n^t \\ D_n & 0 \end{pmatrix} \begin{pmatrix} C_n^t \\ \lambda_n^t \end{pmatrix} = \begin{pmatrix} b_n^t \\ \mu^t \end{pmatrix}$$

being

$$\begin{aligned}
A_n &= \left(-\int_a^b (B_i^n)'(x)\,(B_j^n)'(x)dx \right)_{\substack{1\le i\le n+3 \\ 1\le j\le n+3}};\\
D_n &= \left(B_j^1(a_i) \right)_{\substack{i=1,2 \\ j=1,\ldots,n+3}} \quad \text{with } a_1 = a,\ a_2 = b;\\
C_n &= (c_1, \ldots, c_{n+3});\\
\lambda_n &= \left(\lambda_1^n, \lambda_2^n\right);\\
b_n &= \left(\int_a^b f(x, \sigma_{n-1}(x))\, B_j^n(x)dx \right)_{1\le j\le n+3};\\
\mu &= (\alpha, \beta).
\end{aligned}$$

4 Convergence result

In order to prove that the constructed sequence of approximate functions $(\sigma_n)_{n\in\mathbb{N}}$ converges to the solution of problem (1), we need to introduce the following results.

Firstly, let us consider the subsets

$$\begin{aligned}
H &= \{v \in S_3(T_n) \mid v(a) = \alpha, v(b) = \beta\},\\
H_0 &= \{v \in S_3(T_n) \mid v(a) = v(b) = 0\}.
\end{aligned}$$

Lemma 1 *We assume that the sequence $\sigma_{n-1} \in S_3(T_n)$ has been constructed. Then there exists a unique $\sigma_n \in H$ such that $J_n(\sigma_n) \le J_n(v)$, for all $v \in H$, being J_n the functional defined from $H^1(I)$ into $\mathbb{R}$ by*

$$J_n(v) = \int_a^b v'(x)^2 dx + 2\int_a^b f(x, \sigma_{n-1}(x))v(x)dx.$$

Moreover σ_n verifies for all $v \in H_0$

$$\int_a^b \sigma_n'(x)v'(x)dx = -\int_a^b f(x, \sigma_{n-1}(x))v(x)dx. \tag{3}$$

Proof Let us consider the bilinear form $\tilde{a}$ defined form $S_3(T_n) \times S_3(T_n)$ into $\mathbb{R}$ by

$$\tilde{a}(u, v) = 2(u, v)_1 + 2u(a)v(a) + 2u(b)v(b).$$

It is obvious that $\tilde{a}$ is symmetric, continuous and endowed with a norm in $S_3(T_n)$ defined by $[[v]] = \tilde{a}(v, v)^{1/2}$, which is equivalent to the Sobolev's norm $\|\cdot\|$ in $S_3(T_n) \subset H^1(I)$. Hence, $\tilde{a}$ is a $H^1(I)$-elliptic and the subset H is convex, bounded and not empty.

Now, the application defined by

$$\varphi(v) = -2\int_a^b f(x, \sigma_{n-1}(x))v(x)dx$$

is linear and continuous in $S_3(T_n)$. So, by applying the Lax–Milgram theorem, we deduce that there exists a unique $\sigma_n \in H$ such that $\tilde{a}(\sigma_n, v) = \varphi(v)$, for all $v \in H_0$.

Furthermore, σ_n is characterized to be the minimum in H of the functional $\Phi(v) = \frac{1}{2}\tilde{a}(v, v) - \varphi(v)$ which, in turn, is equivalent to minimizes in H the functional J_n given by

$$J_n(v) = \Phi(v) - v(a)^2 - v(b)^2.$$

□

Lemma 2 *There exists a unique* $(\sigma_n, \lambda_1, \lambda_2) \in H \times \mathbb{R}^2$ *verifying (2), being* σ_n *the solution of (3).*

Proof Let $\{\varphi_1, \ldots, \varphi_{n+1}\}$ be the Lagrange basis functions of $S_3(T_n)$ associated with the functionals $\{\Phi_1, \ldots, \Phi_{n+1}\}$ defined by $\Phi_i(v) = v(x_{i-1})$, for $i = 1\ldots, n+1$, $\Phi_{n+2}(v) = v''(a)$, $\Phi_{n+3}(v) = v''(b)$.

Given $u \in S_3(T_n)$ we define

$$w = u - u(a)\varphi_1 - u(b)\varphi_{n+1},$$

then,

$$\begin{aligned} w(a) &= u(a) - u(a)\varphi_1(a) - u(b)\varphi_{n+1}(a) = 0,\\ w(b) &= u(b) - u(a)\varphi_1(b) - u(b)\varphi_{n+1}(b) = 0, \end{aligned}$$

because $\varphi_1(a) = \varphi_{n+1}(b) = 1$ and $\varphi_1(b) = \varphi_{n+1}(a) = 0$.

Hence, we deduce that $w \in H_0$ and using (3), one has

$$\int_a^b \sigma_n'(x)w'(x)dx = -\int_a^b f(x, \sigma_{n-1}(x))w(x)dx,$$

with σ_n is the solution of (3), and we obtain

$$\begin{aligned} &\int_a^b \sigma_n'(x)u'(x)dx - \left(\int_a^b \sigma_n'(x)\varphi_1'(x)dx\right)v(a) - \left(\int_a^b \sigma_n'(x)\varphi_{n+1}'(x)dx\right)v(b)\\ &= -\int_a^b f(x, \sigma_{n-1}(x))u(x)dx + \left(\int_a^b f(x, \sigma_{n-1}(x))\varphi_1(x)dx\right)v(a)\\ &\quad + \left(\int_a^b f(x, \sigma_{n-1}(x))\varphi_{n-1}(x)dx\right)v(b). \end{aligned}$$

Now, let consider

$$\lambda_1^n = -\int_a^b \left(f(x, \sigma_{n-1}(x))\varphi_1(x) + \sigma_n'(x)\varphi_1'(x)\right) dx$$
$$\lambda_2^n = -\int_a^b \left(f(x, \sigma_{n-1}(x))\varphi_{n+1}(x) + \sigma_n'(x)\varphi_{n+1}'(x)\right) dx$$

hence, we obtain (2). The uniqueness can be readily obtained. □

Theorem 1 *If σ is the solution of (1), then it is verified that*

$$\lim_{n\to+\infty} \|\sigma_n - \sigma\| = 0.$$

Proof (1) Let $h = \dfrac{b-a}{n}$ and S_n be the interpolating spline of σ in $S_3^2(T_n)$ such that $S_n''(a) = S_n''(b) = 0$, then we have that (see [9])

$$\|\sigma - S_n\| \leq C\, h^{1/2}. \tag{4}$$

Hence, being σ_n the minimum of J_n in H we have that $J_n(\sigma_n) \leq J_n(S_n)$ and it follows

$$|\sigma_n|_1^2 + 2(F\sigma_{n-1}, \sigma_n)_0 \leq |S_n|_1^2 + 2(F\sigma_{n-1}, S_n)_0$$

Then

$$|\sigma_n|_1^2 \leq |S_n|_1^2 + 2(F\sigma_{n-1}, S_n - \sigma_n)_0 \tag{5}$$

From (3), taking into account that $\sigma_n - S_n \in H_0$, one can obtain

$$(\sigma_n, \sigma_n - S_n)_1 = -(f(., \sigma_{n-1}(.)), \sigma_n - S_n)_0$$

and, using (5), we deduce that

$$|\sigma_n|_1^2 \leq |S_n|_1^2 + 2(\sigma_n, S_n - \sigma_n)_1 \tag{6}$$

On the other hand, it is

$$|\sigma_n|_1^2 - |S_n|_1^2 \leq |\sigma_n - S_n|_1^2 = |\sigma_n|_1^2 - 2(\sigma_n, S_n)_1 + |S_n|_1^2$$

and, from (6), we have that

$$|\sigma_n|_1^2 \leq |S_n|_1^2 + |S_n|_1^2 + 2(\sigma_n, S_n - \sigma_n)_1 - 2(\sigma_n, S_n)_1 + |S_n|_1^2 = 3|S_n|_1^2 - 2|\sigma_n|_1^2$$

we conclude that

$$|\sigma_n|_1^2 \leq |S_n|_1^2. \tag{7}$$

Then, from (4) and Lemma 1, it follows that there exists $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\|\sigma_n\| \leq C.$$

This means that the family $(\sigma_n)_{n\in\mathbb{N}}$ with $n \geq n_0$ is bounded in $H^1(I)$. Hence, there exists a subsequence $(\sigma_{n_l})_{l\in\mathbb{N}}$ extracted from such a family, with $\lim\limits_{l\to+\infty} n_l = +\infty$ and an element $\sigma^* \in H^1(I)$ such that

$$\sigma^* = \lim_{l\to+\infty} \sigma_{n_l} \quad \text{weakly in } H^1(I). \tag{8}$$

(2) Let us see that $\sigma^* = \sigma$, reasoning by reduction to the absurd. In fact, we suppose that $\sigma^* \neq \sigma$.
For the continuous injection of $H^1(I)$ into $C^0([a, b])$, there exist $\gamma > 0$ and a non-empty open set $W \subset I$ such that

$$\forall x \in W, |\sigma^*(x) - \sigma(x)| > \gamma,$$

and as such injection is also compact, then from (8) it follows that

$$\exists l_0 \in \mathbb{N}, \forall l \geq l_0, \forall x \in W, |\sigma^*(x) - \sigma_{n_l}(x)| \leq \frac{\gamma}{2}\cdot$$

Hence, for all $l \geq l_0$ and all $x \in W$, we have that

$$|\sigma_{n_l}(x) - \sigma(x)| \geq |\sigma^*(x) - \sigma(x)| - |\sigma_{n_l}(x) - \sigma^*(x)| > \frac{\gamma}{2}\cdot \tag{9}$$

We have that $h \to 0$ as $n \to +\infty$, we deduce that, for sufficiently great l, there exists a point $x_0 \in T_n \cap W$ and hence $\sigma_{n_l}(x_0) = \sigma(x_0)$, which is a contradiction with (9).

(3) Given that $\sigma^* = \sigma$ and as $H^1(I)$ is compactly injected in the Sobelev space $H^0(I)$, we have that

$$\sigma = \lim_{l\to+\infty} \sigma_{n_l} \text{ in } H^0(I),$$

and it follows that

$$\lim_{l\to+\infty} |\sigma_{n_l} - \sigma|_0 = 0. \tag{10}$$

So, again using (8) and with $\sigma^* = \sigma$, we have that

$$\lim_{l\to+\infty} (\sigma_{n_l}, \sigma)_1 = \lim_{l\to+\infty} \big(((\sigma_{n_l}, \sigma)) - (\sigma_{n_l}, \sigma)_0\big) = |\sigma|_1^2, \tag{11}$$

with $((v,v)) = \sum_{i=0}^{1}(v,v)_i$ for all $v \in H^1(I)$, stands the inner product associated with the norm $\|\cdot\|$ in $H^1(I)$.
Consequently, given that for all $l \in \mathbb{N}$,

$$\left|\sigma_{n_l} - \sigma\right|_1^2 = \left|\sigma_{n_l}\right|_1^2 + |\sigma|_1^2 - 2(\sigma_{n_l},\sigma)_1, \tag{12}$$

and, from (4) and (7), we have that

$$|\sigma_{n_l}|_1^2 \leq O(h) + |\sigma|_1^2.$$

Then, from (12), it follows that

$$|\sigma_{n_l} - \sigma|_1^2 \leq \mathrm{O}(h) + 2|\sigma|_1^2 - 2|\sigma|_1^2.$$

Hence, using (10), we obtain that

$$\lim_{l\to+\infty} \|\sigma_{n_l} - \sigma\| = 0. \tag{13}$$

4) We conclude, finally, reasoning by reduction to the absurd, that the result of this theorem is satisfied. In fact, if it is not so, there would exist a real number $\gamma > 0$ and a sequence $(n_{l'})_{l'\in\mathbb{N}}$ with $\lim_{l'\to+\infty} n_{l'} = +\infty$ such that

$$\forall l' \in \mathbb{N},\ \left\|\sigma_{n_{l'}} - \sigma\right\| \geq \gamma. \tag{14}$$

Now then, because the sequence $(\sigma_{n_{l'}})_{l'\in\mathbb{N}}$ is bounded in $H^1(I)$, we would deduce, by following the same way as in the points (1), (2) and (3), that from such a sequence a subsequence converging to the function σ can be extracted, which is a contradiction with (14). In short, the result is verified. □

5 Graphical and numerical examples

The goal of this section is to apply our method in order to compute some approximation of the solution of Troesch's problem. To this end, we consider the Troesch's problem:

$$u'' = \lambda \sinh(\lambda\, u),\ 0 \leq x \leq 1,$$

with boundary conditions

$$u(0) = 0,\ u(1) = 1.$$

The expression of the solution of the previous problem is given by (see [11])

$$\sigma(x) = \frac{2}{\lambda}\sinh^{-1}\left[\frac{\sigma'(0)}{2}\, sc\left(\lambda x|1 - \frac{1}{4}\sigma'(0)^2\right)\right]$$

being sc the Jacobi elliptic function. It is deduced that $\sigma'(0) = 2(1-m)^{1/2}$, being m the solution of the equation

$$\frac{\sinh(\frac{\lambda}{2})}{(1-m)^{1/2}} - sc(\lambda|m) = 0.$$

By applying the method studied in sect. 3, we construct a sequence of B-spline functions σ_n, with $n \in \mathbb{N}$, approximating the exact solution of the problem (1) for $f(x, u(x)) = \lambda \sinh(\lambda u(x))$ with $x \in [0, 1]$ and $\lambda = 1$, depending on the number of knots.

Hence, for $n = 2$ let σ be the solution of the problem (1) and σ_2 be its approximating B-spline. The graphs of the curves defined by σ and σ_2 appear in Fig. 1.

For $n = 5$, $n = 9$, $n = 17$, $n = 33$, we have computed σ_5, σ_9, σ_{17} and σ_{33}, respectively.

Now, for $n = 65$ let σ be the solution of the problem (1) and σ_{65} be its approximating B-spline. The graphs of the curves defined by σ and σ_{65} appear in Fig. 2.

Table 1 shows the estimation of the mean square error, designed by E_m, between the solution of the boundary value problem and its approximation by a B-spline function from a partition of n equal intervals of [0, 1], for $n = 2, 3, 5, 9, 17, 33, 65$. The expression of E_m is as follows

$$E_m = \sqrt{\frac{\sum_{i=1}^{1000} (\sigma(a_i) - \sigma_n(a_i))^2}{1000}},$$

where a_i, for i=1,...,1000, are random points of I.

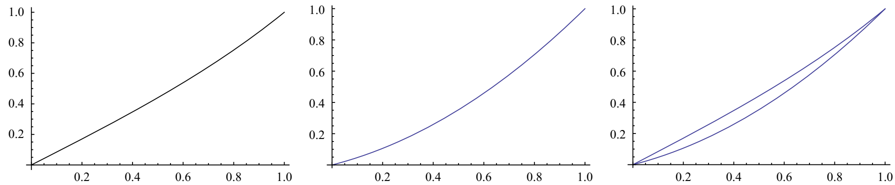

Fig. 1 For $n = 2$. Graph of the *curve* defined by σ, that is the solution of the boundary value problem, the graph of the *curve* defined by a variational B-spline σ_2, that is the approximation of σ and the graph of both curves (from *left* to *right*)

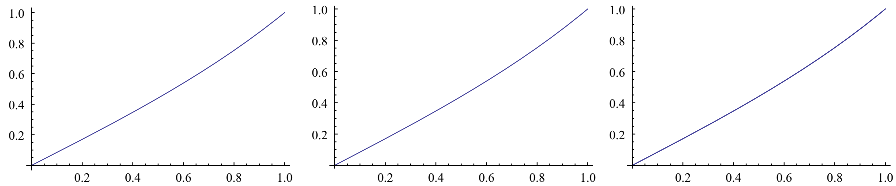

Fig. 2 For $n = 65$. Graph of the *curve* defined by σ, that is the solution of the boundary value problem, the graph of the *curve* defined by a variational B-spline σ_{65}, that is the approximation of σ and the graph of both curves (from *left* to *right*)

Table 1 Table of the estimation of the relative error between the solution of the boundary value problem and its approximation by a B-spline function from a partition of n equal intervals of [0,1]

Number of the knots: n	E_m
2	6.39449×10^{-2}
3	7.05895×10^{-3}
5	8.01515×10^{-4}
9	9.10052×10^{-5}
17	1.03386×10^{-5}
33	1.17531×10^{-6}
65	1.33558×10^{-7}

Table 2 Table of the values of the exact function σ, the constructed approximate function σ_n and the absolute error in each x, for $x = 0.1, \ldots, x = 0.9$

x	$\sigma(x)$	$\sigma_n(x)$	$\lvert error(x) \rvert$
0.1	0.0846612	0.0846569	4.3087×10^{-6}
0.2	0.170171	0.170163	8.2374×10^{-6}
0.3	0.257394	0.257382	0.0000114304
0.4	0.347223	0.347209	0.0000135755
0.5	0.4406	0.440585	0.0000144522
0.6	0.538534	0.53852	0.0000139102
0.7	0.642129	0.642117	0.0000119684
0.8	0.752608	0.752599	8.67314×10^{-6}
0.9	0.871363	0.871358	4.44185×10^{-6}

To compare our method with the existing ones in the literature, we have chosen the paper [3]. Firstly, the solution given in [3] with the collocation method is a discrete solution, while our solution is a continuous function. Secondly, one can observe that the computation of the error in Table 3 of [3] for $\lambda = 1$ is of the order 10^{-3}.

In order to provide a logical comparison between our method and the method presented in [3], we have calculated Table 2 taking the same data as Table 1 (for $\lambda = 1$) in [3], we obtained some better computed values of the error presented in Table 2, reaching the order 10^{-6}.

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